

Analysis of Sun/Moon Gravitational Redshift tests with the STE-QUEST Space Mission

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Abstract. The STE-QUEST space mission will perform tests of the gravitational redshift in the field of the Sun and the Moon to high precision by frequency comparisons of clocks attached to the ground and separated by intercontinental distances. In the absence of Einstein equivalence principle (EP) violation, the redshift is zero up to small tidal corrections as the Earth is freely falling in the field of the Sun and Moon. Such tests are thus null tests, allowing to bound possible violations of the EP. Here we analyze the Sun/Moon redshift tests using a generic EP violating theoretical framework, with clocks minimally modelled as two-level atoms. We present a complete derivation of the redshift (including both GR and non-GR terms) in a realistic experiment such as the one envisaged for STE-QUEST. We point out and correct an error in previous formalisms linked to the atom's recoil not being properly taken into account.

1. Introduction

It is well known that two clocks fixed on the Earth's surface, when compared to each other, do not display a frequency difference due to external masses (Sun, Moon, Planets) at first order in $\Delta U_{\text{ext}}/c^2$ — a fact sometimes referred to as the “*absence of the Noon-Midnight redshift*” [1, 2]. Here U_{ext} is the Newtonian potential of external masses, and c is the speed of light in vacuum. This is of course a direct consequence of the equivalence principle of General Relativity (GR), as the Earth is freely falling in the field of external masses, and there is no effect due to the rotation of the Earth in a local inertial frame. Thus a co-moving frame is inertial and physics is described in that frame by the laws of Special Relativity.

Instead, only tidal terms can be observed, which are a factor $\sim r_E/R_\oplus$ smaller, where r_E is the Earth radius, and R_\oplus is the distance between the Earth centre and the barycentre of the external masses. Hence in the case of the Sun the redshift scales like $\sim GM_\odot r_E^2/(R_\oplus^3 c^2)$ and is very small indeed. For clocks on the Earth's surface such

tidal terms do not exceed a few parts in 10^{17} in fractional frequency, and are thus barely measurable with today's best clocks, whilst first order terms due to the Sun, for example, would be about 5 orders of magnitude larger.

In 2010 the “Space-Time Explorer and QUantum Equivalence principle Space Test” (STE-QUEST) project [3] was proposed to the European Space Agency,[‡] which included a test of the gravitational redshift in the field of the Sun/Moon at first order in $\Delta U_{\text{ext}}/c^2$. The idea is to compare, *via* the STE-QUEST satellite, two clocks attached to the Earth and separated by intercontinental distances. The comparison is made using microwave links in common-view mode. The test boils down to a search for a periodic signal with known frequency and phase in the clock comparison data. Roughly speaking the proposed experiment would test the gravitational redshift from external masses at a level of $\sim (\Delta f/f)(c^2/\Delta U_{\text{ext}})$, where $\Delta f/f$ is the uncertainty in fractional frequency of the used clocks and frequency transfer techniques.

During the assessment study of STE-QUEST [4], and on several subsequent occasions, it was argued that such a test was impossible because of the absence of Noon-Midnight redshift — there is no Sun/Moon redshift to measure at first order in $\Delta U_{\text{ext}}/c^2$, instead only tidal terms can be measured, and therefore the proposed test would perform less well, roughly by a factor $\sim r_E/R_\oplus$.

In this paper we show that such claims are unfounded. Indeed, the conclusion is only valid when the equivalence principle (EP) is satisfied. As tests of the gravitational redshift are precisely testing that hypothesis, *i.e.* consider a possible violation of the EP, the absence of the Noon-Midnight redshift cannot be taken for granted in that situation. More generally, such tests search for a general anomalous coupling between the clock internal energy and the source of the gravitational field (Sun, Moon, ...), couplings which would lead precisely to slight deviations from the null-redshift due to external masses. We show within a simple but general theoretical framework [5, 6, 7, 8] that such deviations are proportional directly to $\Delta U_{\text{ext}}/c^2$, and not only to tidal terms. At this occasion we point out and correct an error in the previous formalism of Ref. [7] (see also [8]), in which the atom's recoil velocity was not properly taken into account, resulting in a wrong sign for the second-order Doppler effect in the redshift formula.

More generally, we believe that our results will be of interest to anyone interested in experimental tests of the equivalence principle, as they present a simple and straightforward way of modelling such experiments and evaluating their respective merits and underlying theoretical connections. We also take this opportunity to present a general and straightforward derivation of the standard GR result.

For simplicity our derivations will be restricted to a rather elementary (quasi-Newtonian, with almost no quantum mechanics involved) Lagrangian-based formalism, which is directly issued from Refs. [5, 6, 7, 8]. However we expect that our conclusions will remain unchanged when working within the very broad and sophisticated Standard Model Extension (SME) formalism [9, 10, 11], as well as other EEP violating frameworks

[‡] STE-QUEST is currently competing for the ESA medium size mission call M5.

such as [12, 13]. A full analysis of STE-QUEST in those frameworks is beyond the scope of this paper, but will be subject of future work.

Our article is organized as follows: We first briefly describe in Sec. 2 the EP violating theoretical framework to be used. Then we apply it in Sec. 3 to a highly simplified “Gedanken” experiment that allows us to derive our main conclusion in a few simple steps. This section can be skipped by readers that are interested only in a fully realistic and complete scenario. Such a scenario is treated in Sec. 4 with a complete calculation for a realistic experiment as envisaged, for example, in the STE-QUEST project, including all relevant terms, both for the standard GR contribution and for the EP-violating terms. We finish with some concluding remarks in Sec. 5.

2. Theoretical framework

In this paper we shall perform our analysis using a broad class of equivalence principle (EP) violating frameworks, encompassing a large class of non-metric gravity theories and consistent with Schiff’s conjecture [14]. This class of theories is defined by a modified Lagrangian describing some physical composite system (*e.g.* an atom or an atomic clock), in which the coupling between gravitation and different types of internal mass-energies is generically not universal, *i.e.* depends on the system and the type of energy in question. Assimilating the system to a point mass in the gravitational field, and working in the non relativistic approximation, we have the Lagrangian

$$L = -mc^2 + \frac{1}{2}m \mathbf{V}^2 + m U(\mathbf{X}), \quad (1)$$

where m is the total mass of the system, $U \equiv U_{\text{ext}}$ is the Newtonian gravitational potential at the position $\mathbf{X} = (X^i)$ in a global frame, *e.g.* centred on the Solar System barycenter, and $\mathbf{V} = (V^i)$ is the velocity of the system in that frame. Following the “Modified Lagrangian framework” (slightly adapted from Refs. [5, 6, 7, 8]) the violation of the EP is encoded into an abnormal dependence of the mass of the system on the velocity \mathbf{V} and position \mathbf{X} . This implies a violation, respectively, of the local Lorentz invariance (LLI) and local position invariance (LPI) aspects of the Einstein equivalence principle [5, 6, 7, 8]. Such abnormal dependence is due to a particular internal energy E_X in the system, hence

$$m(\mathbf{X}, \mathbf{V}) = \bar{m} + \frac{1}{c^2} \left[E_X(\mathbf{X}, \mathbf{V}) + \sum_{Y \neq X} \bar{E}_Y \right], \quad (2)$$

while the other types of energies \bar{E}_Y behave normally. Here \bar{m} is the sum of the rest masses of the particles constituting the system. We shall pose

$$E_X(\mathbf{X}, \mathbf{V}) = \bar{E}_X - \frac{1}{2} \delta m_{\text{I}}^{ij} V^i V^j - \delta m_{\text{P}}^{ij} U^{ij}(\mathbf{X}), \quad (3)$$

where U^{ij} denotes the usual Newtonian tensor potential,[§] and δm_{I}^{ij} and δm_{P}^{ij} are two constant tensors describing the EP violation. The sum of all ordinary energies in

[§] Thus, $U^{ij}(\mathbf{X}) \equiv G \int d^3x \rho(\mathbf{x}) \frac{(X-x)^i (X-x)^j}{|\mathbf{X}-\mathbf{x}|^3}$, whose trace is $U^{ii} = U$.

the system is then $\bar{E} = \bar{E}_X + \sum_{Y \neq X} \bar{E}_Y$, and we may define $m_0 = \bar{m} + \bar{E}/c^2$. The Lagrangian (1) now becomes

$$L = -m_0 c^2 + \frac{1}{2} m_0 (\delta^{ij} + \beta_I^{ij}) V^i V^j + m_0 (\delta^{ij} + \beta_P^{ij}) U^{ij}(\mathbf{X}), \quad (4)$$

where we have neglected small higher-order relativistic corrections. For later convenience we defined two LLI and LPI violating parameters β_I^{ij} and β_P^{ij} by

$$\beta_I^{ij} = \frac{\delta m_I^{ij}}{m_0}, \quad \beta_P^{ij} = \frac{\delta m_P^{ij}}{m_0}. \quad (5)$$

Obviously these parameters depend on the system under consideration.

By varying the Lagrangian (4) we obtain the equation of motion of the system as

$$\frac{dV^i}{dt} = (\delta^{ij} - \beta_I^{ij}) \frac{\partial U}{\partial X^j} + \beta_P^{jk} \frac{\partial U^{jk}}{\partial X^i}. \quad (6)$$

The system does not obey the weak equivalence principle (WEP) and the parameters (5) can also be seen as WEP violating parameters, respectively modifying the inertial (I) and passive (P) gravitational masses of the system. Of course this means that the three different aspects of the equivalence principle (WEP, LLI and LPI) are entangled together by Schiff's conjecture [14]. Finally the energy and linear momentum of the system in the Newtonian approximation read

$$E = m_0 c^2 + \frac{1}{2} m_0 (\delta^{ij} + \beta_I^{ij}) V^i V^j - m_0 (\delta^{ij} + \beta_P^{ij}) U^{ij}(\mathbf{X}), \quad (7)$$

$$P^i = m_0 (\delta^{ij} + \beta_I^{ij}) V^j. \quad (8)$$

In the following we shall apply the modified Lagrangian (4) to the case of an atomic clock moving in the gravitational field of the Sun or the Moon. The clock will be modelled by a two-level atom, and of course we shall assume that the energy responsible for the transition between levels in the atom is the abnormal internal energy E_X .

3. A simple Gedanken experiment

In this section we examine a highly idealized situation in order to simplify the calculations as much as possible, whilst allowing the derivation of our main result. A full and general analysis is presented in Sec. 4. We consider a perfectly spherical Earth in circular orbit around a perfectly spherical Sun, with two identical clocks fixed to the Earth's surface. We work throughout in a Sun-centred, non rotating coordinate system. The clocks are two level atoms, and a typical redshift experiment is modelled as follows (see Fig. 1): Clock *A* undergoes a transition from its excited state $|e\rangle$ to its ground state $|g\rangle$ and emits a photon toward clock *B*. As the photon passes *B*, clock *B* undergoes a transition and emits a photon in the same direction as the one incident from *A*. An observer then measures the frequency difference between the two photons (*cf.* [7]). For simplicity in this section, we assume that the photon takes a direct path from *A* to *B* through the Earth, that the atomic transitions are two-photon transitions^{||} with the two

^{||} Two-photon transitions involve initial and final atomic states whose difference in angular momentum is such that its conservation requires the absorption/emission of two photons [15].

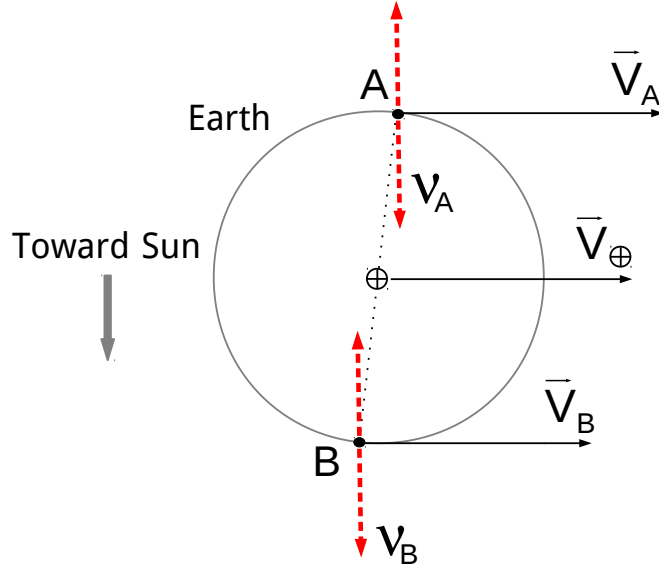


Figure 1. Simplified experimental principle. Two photons ν_A are emitted from clock A and compared to photons ν_B emitted from clock B . The velocities of A and B are perpendicular to the directions of emission of the photons. All positions and velocities are given in the Sun centred frame.

photons emitted in opposite directions, and that the experiment is arranged in such a way that the velocities of A and B (in the Sun-centred frame) are perpendicular to the directions of the respective emitted photons (see Fig. 1).

Furthermore we will use a minimal form of the formalism described in Sec. 2, with $\delta m_1^{ij} = 0$ and $\delta m_P^{ij} = \delta m_P \delta^{ij}$, so the Lagrangian (4) is simplified to

$$L = -m_0 c^2 + \frac{1}{2} m_0 V^2 + (m_0 + \delta m_P) U, \quad (9)$$

where $U = U(\mathbf{X})$ is the usual scalar Newtonian gravitational potential.

3.1. The photon frequency

We calculate the coordinate frequency ν of an emitted photon from energy and momentum conservation between the initial and final states. We write, to lowest order, the initial and final energies and momenta of the system (atom + photons) in the Sun centred frame, using (9) and the atom's conserved quantities (7)–(8),

$$E_f = m_0^g c^2 + \frac{1}{2} m_0^g V_f^2 - (m_0^g + \delta m_P^g) U + 2h\nu, \quad (10)$$

$$E_i = m_0^e c^2 + \frac{1}{2} m_0^e V_i^2 - (m_0^e + \delta m_P^e) U, \quad (11)$$

where h is Planck's constant, together with

$$P_f = m_0^g V_f, \quad P_i = m_0^e V_i. \quad (12)$$

We recall that $m_0^e = \bar{m} + \bar{E}_e/c^2$ and $m_0^g = \bar{m} + \bar{E}_g/c^2$, with δm_P^e and δm_P^g the abnormally coupled energy contributions to the excited and ground states.

Momentum conservation yields to leading order $V_f = (1 + \frac{\Delta \bar{E}}{\bar{m}c^2})V_i$ with $\Delta \bar{E} = \bar{E}_e - \bar{E}_g$. Note that because of the two-photon process the direction of V is unchanged. To leading order energy conservation then leads to

$$h\nu = h\nu_0 \left(1 - \frac{V_i^2}{2c^2} - \frac{U}{c^2} (1 + \alpha_P) \right), \quad (13)$$

where we pose $2h\nu_0 = \Delta \bar{E}$ for the two-photon process, $\alpha_P = c^2 \Delta \delta m_P / \Delta \bar{E}$ and $\Delta \delta m_P = \delta m_P^e - \delta m_P^g$.

It is interesting to note that the initial and final velocities of the atom are not equal. Indeed, this is required by momentum conservation as the initial and final masses are unequal. The change of velocity is directly related to aberration, *i.e.* although the photon directions are perpendicular to the atom's velocity in the Sun centred frame, they are not perpendicular in a frame at constant velocity V_i , thus in that frame the atom receives a recoil “kick”, with a corresponding change of velocity in the Sun centred frame (of course, the additional velocity is quickly dissipated in the clock structure). Alternatively, one can arrange the experiment such that the photons are emitted at a slight angle (*i.e.* $\theta \sim V_i/c$ in radians) in the Sun centred frame corresponding to perpendicular emission in the V_i frame. In that case the atom velocity is unchanged. The corresponding calculation leads to the same result as (13).

3.2. Derivation of the redshift

We calculate the coordinate frequency difference $(\nu_A - \nu_B)/\nu_0$ of the two co-located photons emitted at A and B , respectively (*cf.* Fig. 1). Applying Eq. (13) to the emitted photons by A and B gives directly

$$\frac{\nu_A - \nu_B}{\nu_0} = \frac{U_B - U_A}{c^2} (1 + \alpha_P) + \frac{V_B^2 - V_A^2}{2c^2}. \quad (14)$$

For simplicity, we will assume that the Earth gravitational potential is the same at A and B and we thus neglect it in the difference $U_B - U_A$, and we also neglect small correction terms in $V_B^2 - V_A^2$ due to the rotation of the Earth (see the next section for a fuller treatment). The angular frequency of the Earth in circular orbit is $\Omega^2 = (1 + \beta_P)GM_\odot/R_\oplus^3$, with M_\odot the (reduced) mass of the Sun, $R_\oplus = |\mathbf{X}_\oplus|$ the Earth-Sun distance, and $\beta_P = \delta m_P/m_0$ for the WEP violating parameter in the Sun field. The potentials are $U_{A,B} = \frac{GM_\odot}{R_{A,B}}$ and the velocities $V_{A,B}^2 = (\Omega R_{A,B})^2$. Then the frequency difference (14) becomes

$$\frac{\nu_A - \nu_B}{\nu_0} = \frac{GM_\odot}{c^2} \left(\frac{1}{R_B} - \frac{1}{R_A} \right) (1 + \alpha_P) + \frac{GM_\odot}{2c^2 R_\oplus^3} (R_B^2 - R_A^2) (1 + \beta_P). \quad (15)$$

Writing $R_{A,B} = R_\oplus + \Delta R_{A,B}$ and expanding in terms of the small quantities $\Delta R_{A,B}/R_\oplus$ we finally obtain

$$\frac{\nu_A - \nu_B}{\nu_0} = \frac{GM_\odot}{c^2} \left(\frac{1}{R_B} - \frac{1}{R_A} \right) \alpha_P + \mathcal{O} \left(\frac{GM_\odot}{R_\oplus c^2} \frac{\Delta R_{A,B}^2}{R_\oplus^2} \right), \quad (16)$$

where we have neglected terms which are suppressed by a factor $\beta_P/\alpha_P \lesssim \Delta\bar{E}/(m_{\text{at}}c^2) \ll 1$, with m_{at} the average atomic mass of the elements making up the Earth [see Eq. (21)]. When the equivalence principle is satisfied ($\alpha_P = 0$) one recovers the expected result [1, 2] that the frequency difference is zero, up to tidal terms. However, in presence of a violation of the equivalence principle a frequency difference remains *at first order in* $\Delta U/c^2$, which is therefore testable experimentally.

4. Full analysis of a realistic experiment

4.1. The photon frequency

We now work out the general case of the full modified Lagrangian (4) without simplifying assumptions regarding the direction of propagation of the photons. We thus consider again the transition between an initial excited $|e\rangle$ state toward a final ground $|g\rangle$ state of the atom, accompanied by the emission of a photon in the unit direction $\mathbf{N} = (N^i)$ as measured in the global (Sun centred) frame. In particular, we now consider a one-photon process. Conservation of energy and linear momentum between the initial and final states implies that

$$\Delta E = h\nu, \quad \Delta \mathbf{P} = \frac{h\nu}{c} \mathbf{N}, \quad (17)$$

where ν denotes the coordinate frequency of the emitted photon, and $\Delta E = E_e - E_g$ and $\Delta \mathbf{P} = \mathbf{P}_e - \mathbf{P}_g$ are the changes in the energy and linear momentum (7)–(8). Then a simple calculation extending the one of Sec. 3, combining both the energy and linear momentum conservation laws, thus taking into account the recoil velocity $\Delta \mathbf{V} = \mathbf{V}_e - \mathbf{V}_g$ of the atom during the transition, gives the frequency of the photon as

$$h\nu = \frac{h\nu_0}{1 - \mathbf{N} \cdot \mathbf{V}/c} \left[1 - \frac{1}{2} (\delta^{ij} + \alpha_1^{ij}) \frac{V^i V^j}{c^2} - (\delta^{ij} + \alpha_P^{ij}) \frac{U^{ij}(\mathbf{X})}{c^2} \right], \quad (18)$$

where we pose $h\nu_0 = \Delta\bar{E}$ with $\Delta\bar{E} = \bar{E}_e - \bar{E}_g$ and we recall that $m_0^e = \bar{m} + \bar{E}_e/c^2$ and $m_0^g = \bar{m} + \bar{E}_g/c^2$. The violation of the universal redshift is here parametrized by the two parameters

$$\alpha_1^{ij} = \frac{\Delta\delta m_1^{ij}}{\Delta\bar{E}/c^2}, \quad \alpha_P^{ij} = \frac{\Delta\delta m_P^{ij}}{\Delta\bar{E}/c^2}. \quad (19)$$

In addition we can compute the recoil velocity $\Delta \mathbf{V}$ as

$$m_0 \Delta V^i = h\nu (\delta^{ij} - \beta_1^{ij}) \frac{N^j}{c} - h\nu_0 (\delta^{ij} + \alpha_1^{ij} - \beta_1^{ij}) \frac{V^j}{c^2}. \quad (20)$$

These results are completely general, valid for a photon emitted in any direction \mathbf{N} , and take in particular into account the recoil of the atom. Notice that the recoil effect has two consequences on the formula (18): (i) It implies the usual first-order Doppler effect in the factor in front of the formula (18), and (ii) it yields the correct sign for the second-order Doppler (or kinetic) term. Even in standard GR (*i.e.* independently from any EP violation), taking properly into account the atom's recoil is crucial in order to obtain the correct sign for the kinetic term in Eq. (18). In that respect the corresponding

formula in Ref. [7], see Eq. (2.22) there, and the formula (2.38) which was replicated in Ref. [8], are incorrect as they do not include the recoil of the atom, yielding the wrong sign for the second-order Doppler effect.

The result (18) gives us the coordinate frequency of the emitted photon depending on the internal physics of the atom, and in particular on the redshift violating parameters (19) associated with possible LLI and LPI violations of some internal energy E_X in the atom. It is well known [6] that the WEP violating parameters (5) are related to their redshift violating counterparts (19) by

$$\beta_I^{ij} \simeq \alpha_I^{ij} \frac{\bar{E}}{m_0 c^2}, \quad \beta_P^{ij} \simeq \alpha_P^{ij} \frac{\bar{E}}{m_0 c^2}, \quad (21)$$

so that WEP and redshift (or LPI/LLI) tests are not independent from each other, but their relative interests depend on the details of the model for the EP violation, and in particular on the type of abnormal energy E_X involved and the type of atom considered. The same conclusion arises also in different formalisms, such as the powerful Standard Model Extension (SME) [9, 10, 11], and in formalisms motivated by string theory involving dilatonic or moduli scalar fields with gravitational strength, whose couplings to matter violate the EEP [12, 13]. (See also further comments in Sec. 5.)

4.2. Transformation to an Earth centred frame

The modified Lagrangian (4) is defined in the global coordinate system (T, \mathbf{X}) associated with the Sun — centred on the Solar System barycenter. In this section we shall need to apply a coordinate transformation to a local coordinate system (t, \mathbf{x}) attached to the Earth, and centred on the center of mass of the Earth.

Let $\mathbf{X}_\oplus(T)$ be the trajectory of the Earth around the Sun in global coordinates. At a particular instant T_0 the position, velocity and acceleration of the Earth are $\mathbf{X}_\oplus \equiv \mathbf{X}_\oplus(T_0)$, $\mathbf{V}_\oplus \equiv \mathbf{V}_\oplus(T_0)$ and $\mathbf{A}_\oplus \equiv \mathbf{A}_\oplus(T_0)$. The acceleration of the Earth at that instant is $\mathbf{A}_\oplus = \nabla U(\mathbf{X}_\oplus)$, where U is the gravitational potential of the Sun and/or other bodies of the Solar System.¶ In a neighbourhood of the particular event (T_0, \mathbf{X}_\oplus) , we *define* the local (accelerated) coordinate system (t, \mathbf{x}) centred on the Earth by

$$t = T - T_0 - \frac{\mathbf{V}_\oplus \cdot (\mathbf{X} - \mathbf{X}_\oplus)}{c^2}, \quad (22)$$

$$\mathbf{x} = \mathbf{X} - \mathbf{X}_\oplus - \mathbf{V}_\oplus (T - T_0) - \frac{1}{2} \mathbf{A}_\oplus (T - T_0)^2. \quad (23)$$

For the present analysis, essentially confined to Newtonian order, such coordinate transformation will be sufficient. In particular we do not need to consider the well-known acceleration term in the temporal transformation law which would read $t = (T - T_0)[1 - (\mathbf{X} - \mathbf{X}_\oplus) \cdot \mathbf{A}_\oplus / c^2 + \dots]$, as well as other terms coming from the Lorentz transformation. Only in Sec. 4.4, where we consider an arbitrary satellite velocity, will

¶ Note that the acceleration of the Earth should include terms in β_I and β_P as in (6). However, in the final result these terms are suppressed by a factor $\beta/\alpha \ll 1$ [*cf.* (21)], and can thus be neglected.

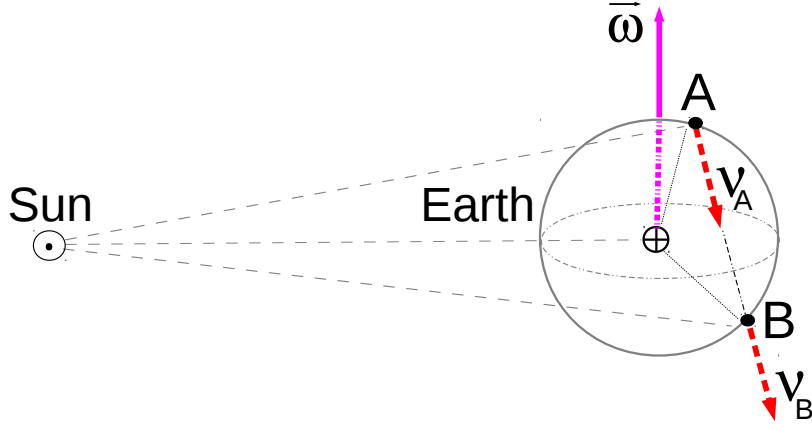


Figure 2. The photon ν_A is emitted from ground clock A toward B and compared to the photon emitted from ground clock B in the same direction as photon ν_A .

a non-Galilean term, the $\mathcal{O}(c^{-2})$ term in (22), be required. This term ensures that the speed of light is an invariant within the required approximation (see Appendix A), and thus allows a correct calculation of the first order Doppler effect in the local frame. Note in this respect that we are assuming that the violation of the equivalence principle affects the internal physics of the atom through the modified Lagrangian (4), but that the law of propagation of photons is “standard”.

Consider the emission of a photon by an atomic transition at the position \mathbf{X}_A and instant T_A in the global frame. This photon propagates from A to B and is received at the position \mathbf{X}_B and instant T_B . Simultaneously with the reception of this photon at T_B , another photon is emitted by the same atomic transition of another atom of the same species at point \mathbf{X}_B in the same direction as the photon coming from A (see Fig. 2). We want to compute the difference of coordinate frequencies ν_A and ν_B of these photons, obtained from Eq. (18).

For convenience we shall choose the instant T_0 to be the average between the emission and reception instants, namely

$$T_0 = \frac{T_A + T_B}{2}. \quad (24)$$

Defining also $\mathbf{R}_{AB} = \mathbf{X}_B(T_B) - \mathbf{X}_A(T_A)$ and the associated unit direction $\mathbf{N}_{AB} = \mathbf{R}_{AB}/R_{AB}$ where $R_{AB} = |\mathbf{R}_{AB}|$, we have $R_{AB} = c(T_B - T_A)$ for the light-like separation. In the local frame the emission and reception events (t_A, \mathbf{x}_A) and (t_B, \mathbf{x}_B) are given by (22)–(23). In particular, with our choice of “central” instant (24), the emission and reception in the local frame occur at

$$t_A = -\frac{R_{AB}}{2c} + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (25)$$

$$t_B = \frac{R_{AB}}{2c} + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (26)$$

that are small quantities. At this stage we do not need to consider the $1/c^2$ correction term in (22). Using this, the spatial positions in the local frame are then deduced from (22)–(23). Neglecting terms $\sim 1/c^2$ we get

$$\mathbf{x}_A = \mathbf{X}_A - \mathbf{X}_\oplus + \frac{R_{AB}}{2c} \mathbf{V}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (27)$$

$$\mathbf{x}_B = \mathbf{X}_B - \mathbf{X}_\oplus - \frac{R_{AB}}{2c} \mathbf{V}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (28)$$

By differentiating (22)–(23) we compute the corresponding velocities as

$$\mathbf{v}_A = \mathbf{V}_A - \mathbf{V}_\oplus + \frac{R_{AB}}{2c} \mathbf{A}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (29)$$

$$\mathbf{v}_B = \mathbf{V}_B - \mathbf{V}_\oplus - \frac{R_{AB}}{2c} \mathbf{A}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (30)$$

The main point of our calculation is to implement the fact that the two emitting and receiving atoms/clocks are attached to the Earth. This assumption will be implemented in the most general way by imposing that *in the local frame* (t, \mathbf{x}) defined by Eqs. (22)–(23) the motion of the atom/clock is that of a rigid rotator, *i.e.*

$$\mathbf{v}_A = \boldsymbol{\omega} \times \mathbf{x}_A, \quad \mathbf{v}_B = \boldsymbol{\omega} \times \mathbf{x}_B, \quad (31)$$

where $\boldsymbol{\omega}$ denotes the rotation vector of the Earth supposed to be constant. Here, consistent with our Newtonian approximation, we only need to consider the usual rigid rotation conditions (31).⁺ Translated into the global frame (T, \mathbf{X}) the latter assumption imposes that

$$\mathbf{V}_A - \mathbf{V}_\oplus = \boldsymbol{\omega} \times (\mathbf{X}_A - \mathbf{X}_\oplus) + \frac{R_{AB}}{2c} [\boldsymbol{\omega} \times \mathbf{V}_\oplus - \mathbf{A}_\oplus] + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (32)$$

$$\mathbf{V}_B - \mathbf{V}_\oplus = \boldsymbol{\omega} \times (\mathbf{X}_B - \mathbf{X}_\oplus) - \frac{R_{AB}}{2c} [\boldsymbol{\omega} \times \mathbf{V}_\oplus - \mathbf{A}_\oplus] + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (33)$$

Subtracting those equations we get for the relative velocity,

$$\mathbf{V}_B - \mathbf{V}_A = \boldsymbol{\omega} \times \mathbf{R}_{AB} - \frac{R_{AB}}{c} [\boldsymbol{\omega} \times \mathbf{V}_\oplus - \mathbf{A}_\oplus] + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (34)$$

4.3. Derivation of the redshift

We now use the previous relations and the result (18) to explicitly compute the frequency shift between the photons A and B . Let us first recover the standard GR prediction, setting α_I^{ij} and α_P^{ij} in (18) to zero. Expanding to the required order we obtain

$$\begin{aligned} \left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}} &= \frac{1}{c} \mathbf{N}_{AB} \cdot (\mathbf{V}_A - \mathbf{V}_B) \\ &+ \frac{1}{c^2} \left[(\mathbf{N}_{AB} \cdot \mathbf{V}_A)^2 - (\mathbf{N}_{AB} \cdot \mathbf{V}_B)^2 - \frac{\mathbf{V}_A^2}{2} - U_A + \frac{\mathbf{V}_B^2}{2} + U_B \right] + \mathcal{O}\left(\frac{1}{c^3}\right). \end{aligned} \quad (35)$$

⁺ See *e.g.* [16] and references therein for discussions on rigidity conditions in special relativity.

Now, thanks to (34), we see that the first term in that expression, which is the usual first order Doppler effect, is actually of second order. We then obtain, in a first stage,*

$$\left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}} = \frac{1}{c^2} \left[U_B - U_A + \frac{\mathbf{V}_B^2}{2} - \frac{\mathbf{V}_A^2}{2} + \mathbf{R}_{AB} \cdot (\boldsymbol{\omega} \times \mathbf{V}_\oplus - \mathbf{A}_\oplus) \right] + \mathcal{O}\left(\frac{1}{c^3}\right). \quad (36)$$

In the small $1/c^2$ terms we can approximate $\mathbf{X}_A = \mathbf{X}_\oplus + \mathbf{x}_A$ and $\mathbf{X}_B = \mathbf{X}_\oplus + \mathbf{x}_B$, and expand the Newtonian potentials U_A and U_B around the value at the center of the Earth, $U_\oplus = U(\mathbf{X}_\oplus)$, with higher order terms of that expansion being the tidal terms. For simplicity we keep only the dominant tidal term at the quadrupole level. Finally the net prediction from GR (and the Einstein equivalence principle) reads as†

$$\left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}} = \frac{1}{2c^2} \left[(x_B^i x_B^j - x_A^i x_A^j) \frac{\partial^2 U}{\partial x^i \partial x^j}(\mathbf{X}_\oplus) + \mathbf{v}_B^2 - \mathbf{v}_A^2 \right] + \mathcal{O}\left(\frac{1}{c^3}\right). \quad (37)$$

Reminiscent of the absence of Noon-Midnight redshift, the latter GR effect is very small, as it scales like $\sim GM_\odot r_E^2 / (R_\oplus^3 c^2)$ and is typically of the order of 10^{-17} . Evidently this is because the Earth is freely falling in the field of the Sun (and of other bodies of the Solar System), so the redshift depends only on the tidal field of the Sun rather than on the field itself. In the freely falling frame the laws of special relativity hold and there is no redshift between clocks. Furthermore the rigid rotation of the Earth does not give rise to an effect either. This can easily be checked in the simple configuration analyzed in Sec. 3. This is due to the fact that photons do not experience any redshift when their emitters and receivers are attached to the rim of a centrifuge in special relativity (see a classic exercise on p. 63 of MTW [17]).

Finally we complete our analysis by simply adding the contributions of the redshift violating parameters α_1^{ij} and α_P^{ij} that are immediately seen from Eq. (18) to result in

$$\frac{\nu_A - \nu_B}{\nu_0} = \left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}} + \frac{1}{c^2} \left[\alpha_1^{ij} \frac{V_B^i V_B^j - V_A^i V_A^j}{2} + \alpha_P^{ij} (U_B^{ij} - U_A^{ij}) \right]. \quad (38)$$

The non-GR terms depend on the velocities $V_{A,B}^i$ in the global frame, since the EP violating parameters α_1^{ij} and α_P^{ij} are defined in that frame, see (18). Note that in the left-hand side of (38) the coordinate frequencies $\nu_{A,B}$ should be the ones measured in the local frame. At that order, because the final effect (38) is already of order $\sim 1/c^2$, we do not need to correct for the coordinate frequencies taking into account the $\mathcal{O}(c^{-2})$ term in the transformation law (22). However, in Sec. 4.4, where we investigate a more realistic case of the comparison of the two clocks A and B *via* a satellite S , we shall need to include that term to transform the frequencies $\nu_{A,B}$.

The same formula applies to any external body in the Solar System, but with the distinction that in the case of a violation of the EP the redshift parameter α_P^{ij} is expected

* Note that $(\mathbf{N}_{AB} \cdot \mathbf{V}_A)^2 - (\mathbf{N}_{AB} \cdot \mathbf{V}_B)^2 = [\mathbf{N}_{AB} \cdot (\mathbf{V}_B - \mathbf{V}_A)][\mathbf{N}_{AB} \cdot (\mathbf{V}_B + \mathbf{V}_A)] = \mathcal{O}\left(\frac{1}{c}\right)$ from (34).

† Intermediate formulas useful in this calculation are

$$\begin{aligned} U_B - U_A &= \frac{1}{2} (x_B^i x_B^j - x_A^i x_A^j) \frac{\partial^2 U}{\partial x^i \partial x^j}(\mathbf{X}_\oplus) + \mathbf{R}_{AB} \cdot \mathbf{A}_\oplus, \\ \frac{\mathbf{V}_B^2}{2} - \frac{\mathbf{V}_A^2}{2} &= \frac{\mathbf{v}_B^2}{2} - \frac{\mathbf{v}_A^2}{2} + (\boldsymbol{\omega} \times \mathbf{R}_{AB}) \cdot \mathbf{V}_\oplus. \end{aligned}$$

to depend on the particular source of the gravitational field (Sun, Moon, *etc.*). Thus its contribution to Eq. (38) should rather be a sum over all possible bodies n with different EP violating parameters $(\alpha_P^{ij})_n$:

$$\frac{\nu_A - \nu_B}{\nu_0} = \left(\frac{\nu_A - \nu_B}{\nu_0} \right)_{\text{GR}} + \frac{1}{c^2} \left[\alpha_1^{ij} \frac{V_B^i V_B^j - V_A^i V_A^j}{2} + \sum_n (\alpha_P^{ij})_n (U_B^{ij} - U_A^{ij})_n \right]. \quad (39)$$

As we can see, despite the fact that the GR contribution is extremely small due to the Earth freely falling toward the Sun, the non-GR LPI violating corrections in (38)–(39) are linear in the potentials of the exterior bodies. This fact allows to test at an interesting level in Earth vicinity the redshift parameters α_1^{ij} and $(\alpha_P^{ij})_n$. For instance, one expects to obtain in the case of the STE-QUEST experiment [3] a test of the gravitational redshift due to the Sun to an uncertainty of $|\alpha_P| \leq 2 \times 10^{-6}$, with an ultimate goal of 5×10^{-7} . For the case of the Moon, the expected uncertainty should be 4×10^{-4} , with an ultimate goal of 9×10^{-5} . Since the measurement will consist in the indirect comparison of signals from two clocks located on the ground *via* the satellite orbiting the Earth, we present in the next subsection a more realistic analysis with a satellite S linked to the two ground clocks A and B .

4.4. Two ground clocks compared via a satellite

The experimental configuration is shown in Fig. 3. The emitter is now on-board the satellite S , and the emitted photons ν_{SA} and ν_{SB} are compared to those emitted by the two ground clocks A and B . The two photons are emitted from the satellite at the same instant, thus all quantities related to the satellite (velocity, gravitational potential, instrumental noise and biases) are the same for both photons. This “common view” arrangement is analogous to the actual situation planned for STE-QUEST, whose orbit is designed precisely to allow for long common view periods between ground clocks located on different continents.

The corresponding frequency difference is calculated from the individual links by

$$\frac{\nu_A - \nu_B}{\nu_0} = \frac{\nu_{SB} - \nu_B}{\nu_0} - \frac{\nu_{SA} - \nu_A}{\nu_0}. \quad (40)$$

The clock on board the satellite is not fixed to the Earth and moves at arbitrary velocity, thus Eq. (39) cannot be directly applied to the individual links as the assumption (31) is not satisfied for S . Instead, going back to (18) and concentrating first on the GR part, we get the same result as in (35),

$$\begin{aligned} \left(\frac{\nu_{SA} - \nu_A}{\nu_0} \right)_{\text{GR}} &= \frac{1}{c} \mathbf{N}_{SA} \cdot (\mathbf{V}_S - \mathbf{V}_A) \\ &+ \frac{1}{c^2} \left[(\mathbf{N}_{SA} \cdot \mathbf{V}_S)^2 - (\mathbf{N}_{SA} \cdot \mathbf{V}_A)^2 - \frac{\mathbf{V}_S^2}{2} - U_S + \frac{\mathbf{V}_A^2}{2} + U_A \right] + \mathcal{O} \left(\frac{1}{c^3} \right), \end{aligned} \quad (41)$$

with a similar expression for the comparison of S to B . We will aim at expressing all quantities in the local frame, in order to recover the standard expressions, in particular for the first order Doppler shifts. This requires converting the photon coordinate

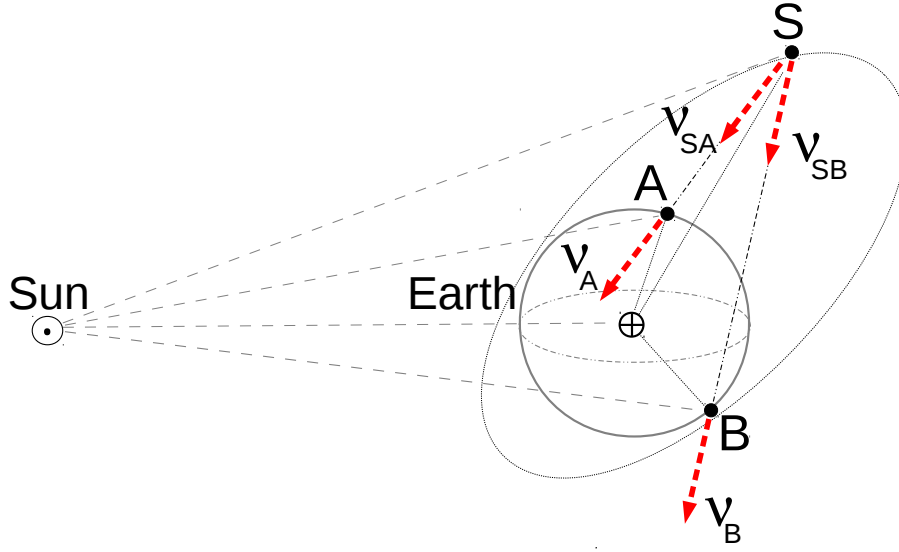


Figure 3. Two photons ν_{SA} and ν_{SB} are emitted from a clock on board a satellite S and compared to photons emitted in the same directions from ground clocks A and B .

frequencies from the global frame, as is used in Eq. (41), to the local frame. As shown in Appendix A the frequency difference transforms as

$$\left(\frac{\nu_{SA} - \nu_A}{\nu_0}\right)_{\text{GR}}^{\text{local}} = \left(\frac{\nu_{SA} - \nu_A}{\nu_0}\right)_{\text{GR}} - \frac{1}{c^2}(\mathbf{n}_{SA} \cdot \mathbf{V}_{\oplus})\left(\mathbf{N}_{SA} \cdot (\mathbf{V}_S - \mathbf{V}_A)\right) + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (42)$$

with a similar expression for the comparison of S to B . Note that with both clocks fixed on the Earth the correction term in (42) is of order $\mathcal{O}(c^{-3})$ thanks to (34), which justifies not having used (42) in the previous sections.

Substituting (41) and (42) into (40), the redshift and second-order Doppler terms due to the satellite cancel leaving

$$\begin{aligned} \left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}}^{\text{local}} &= \frac{1}{c} \left[\mathbf{N}_{SB} \cdot (\mathbf{V}_S - \mathbf{V}_B) - \mathbf{N}_{SA} \cdot (\mathbf{V}_S - \mathbf{V}_A) \right] \\ &+ \frac{1}{c^2} \left[(\mathbf{N}_{SB} \cdot \mathbf{V}_S)^2 - (\mathbf{N}_{SB} \cdot \mathbf{V}_B)^2 - (\mathbf{N}_{SA} \cdot \mathbf{V}_S)^2 + (\mathbf{N}_{SA} \cdot \mathbf{V}_A)^2 \right. \\ &\quad \left. + (\mathbf{n}_{SA} \cdot \mathbf{V}_{\oplus})\left(\mathbf{N}_{SA} \cdot (\mathbf{V}_S - \mathbf{V}_A)\right) - (\mathbf{n}_{SB} \cdot \mathbf{V}_{\oplus})\left(\mathbf{N}_{SB} \cdot (\mathbf{V}_S - \mathbf{V}_B)\right) \right. \\ &\quad \left. + \frac{\mathbf{V}_B^2}{2} + U_B - \frac{\mathbf{V}_A^2}{2} - U_A \right] + \mathcal{O}\left(\frac{1}{c^3}\right). \end{aligned} \quad (43)$$

The first two lines result from (the expansion of) the first order Doppler effect, the third line comes from the transformation (42) and the last line is identical to the corresponding terms in (35). We now use (22)–(23) to transform all \mathbf{N} 's and \mathbf{V} 's to the Earth centred

frame, with $T_0 = T_S$ (the emission time on the satellite) instead of (24).^{††} We also apply the ansatz (31) to the two ground clocks A and B (but not to the satellite clock S). This leads after some vector algebra similar to Sec. 4.3 to a result analogous to (37),

$$\left(\frac{\nu_A - \nu_B}{\nu_0}\right)_{\text{GR}}^{\text{local}} = \Delta_S + \frac{1}{2c^2} \left[(x_B^i x_B^j - x_A^i x_A^j) \partial_{ij} U_{\oplus} + \mathbf{v}_B^2 - \mathbf{v}_A^2 \right] + \mathcal{O}\left(\frac{1}{c^3}\right), \quad (44)$$

with, however, some extra terms representing first-order Doppler effects in the Earth centred frame, which are now non-zero, contrary to Eq. (37), as the satellite clock S is not fixed on the Earth surface. These extra terms are the usual (expansion of) the first order Doppler effect given by

$$\begin{aligned} \Delta_S = & \frac{1}{c} [\mathbf{n}_{SB} \cdot (\mathbf{v}_S - \mathbf{v}_B) - \mathbf{n}_{SA} \cdot (\mathbf{v}_S - \mathbf{v}_A)] \\ & + \frac{1}{c^2} \left[(\mathbf{n}_{SB} \cdot \mathbf{v}_S)^2 - (\mathbf{n}_{SB} \cdot \mathbf{v}_B)^2 - (\mathbf{n}_{SA} \cdot \mathbf{v}_S)^2 + (\mathbf{n}_{SA} \cdot \mathbf{v}_A)^2 \right]. \end{aligned} \quad (45)$$

As expected Eq. (44) only includes tidal terms in the gravitational redshift, which are small. However, the non-GR terms are given exactly like in the previous section by (38) or (39) and can be used to bound non-Einsteinian EP violating parameters at leading order in U/c^2 as previously discussed.

5. Conclusion

We have shown that in a very broad class of equivalence principle violating theories, a test of the gravitational redshift in the field of external bodies (Sun, Moon, ...) using Earth fixed clocks is sensitive *to first order* in $\Delta U_{\text{ext}}/c^2$ to a possible violation. In doing so we have also provided a detailed derivation of the well known result (sometimes coined the absence of Noon-Midnight redshift) that in the GR limit only tidal terms in the gravitational redshift can be observed, which are a factor $\sim r_E/R_{\oplus}$ smaller. The calculations were carried out consistently for different configurations, including the comparison of ground clocks *via* a satellite as planned in the STE-QUEST mission.

We modelled the clocks by two-level atoms and derived the coordinate frequency of the photon emitted by an atomic transition, depending on the internal physics of the atom and in particular on some possible abnormal dependence of the internal energy on the position and velocity of the atom. In the modified Lagrangian formalism [5, 6, 7, 8] (see also [9, 10, 11, 12, 13] for alternative formalisms), such dependence is parametrized by EP violating parameters associated with the violation of the local Lorentz invariance (LLI) and the local position invariance (LPI). We pointed out the importance of taking into account in this formalism the recoil of the atom in order to recover the usual first-order Doppler effect and the correct sign for the second-order Doppler correction.

^{††}Note in particular that, as shown in Appendix B (and similarly for \mathbf{N}_{SB} *vs.* \mathbf{n}_{SB})

$$\mathbf{N}_{SA} = \mathbf{n}_{SA} \left(1 - \frac{1}{c} \mathbf{n}_{SA} \cdot \mathbf{V}_{\oplus} \right) + \frac{1}{c} \mathbf{V}_{\oplus}.$$

In addition to the test of the gravitational redshift, the STE-QUEST satellite will also carry an experiment measuring the weak equivalence principle (WEP) or universality of free fall at the level 10^{-15} by means of dual-species atomic interferometry [3]. Let us thus finish by briefly discussing the relative merits of different types of tests of the equivalence principle in our theoretical framework. As mentioned in Sec. 4.1, tests of the gravitational redshift — LPI aspect of the equivalence principle — as well as tests of LLI are related to tests of WEP *via* a factor $\bar{E}/(m_0 c^2)$, *cf.* (21). Therefore, the comparison between the different tests depends crucially on the details of the chosen model for the violation of the equivalence principle. For example, if all kinds of electromagnetic energy (including e.g. nuclear binding energy) contribute to the equivalence principle violation then $\bar{E}/(m_0 c^2) \approx 10^{-3}$. But if only nuclear spin plays a role then the involved energies are those of the hyperfine transitions (GHz frequencies) and $\bar{E}/(m_0 c^2) \approx 10^{-16}$. Thus WEP and gravitational redshift tests compare differently by many orders of magnitude depending on the detailed model used. For instance, see Ref. [12] for a comparison between WEP and redshift/clock tests using generic dilatonic EP violating models. The most reasonable strategy in a general search for the violation of the equivalence principle is to perform the test of the gravitational redshift alongside with the test of the WEP. This is the driving motivation of the STE-QUEST mission.

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Appendix A. Photon coordinate frequency in the local frame

We work in the geometric optics approximation with the photon's four-vector $K_\mu = \partial\phi/\partial X^\mu = (-\Omega/c_{\text{global}}, \mathbf{K})$ in the global frame (T, \mathbf{X}) , where as usual $\Omega = 2\pi\nu_{\text{global}}$ and $\mathbf{K} = K\mathbf{N}$ with $K = |\mathbf{K}| = \Omega/c_{\text{global}}$. Similarly in the local frame (t, \mathbf{x}) defined by Eqs. (22)–(23), we have $k_\mu = \partial\phi/\partial x^\mu = (-\omega/c_{\text{local}}, \mathbf{k})$ where $\omega = 2\pi\nu_{\text{local}}$ with $\mathbf{k} = k\mathbf{n}$ and $k = \omega/c_{\text{local}}$.

As already mentioned in Sec. 4.2, the extra term $\mathcal{O}(c^{-2})$ in the temporal transformation law (22), where for definiteness we identify $c \equiv c_{\text{global}}$, ensures that the speed of light is invariant to leading order, *i.e.* $c_{\text{local}} = c_{\text{global}}[1 + \mathcal{O}(c^{-2})]$ or, more precisely,

$$c_{\text{local}} = c_{\text{global}} \left[1 + \mathcal{O} \left(\frac{V_\oplus^2}{c^2}, \frac{A_\oplus |\mathbf{X} - \mathbf{X}_\oplus|}{c^2} \right) \right], \quad (\text{A.1})$$

which implies that in the local frame $k = -k_0 = \omega/c_{\text{global}}$ modulo small corrections $\mathcal{O}(c^{-3})$. Using the transformations (22) and (23) we easily find to first order in V_\oplus/c ,

$K_0 = \frac{\partial x^\mu}{\partial X^0} k_\mu = k_0 - \mathbf{k} \cdot \mathbf{V}_\oplus / c$, or equivalently

$$\nu_{\text{global}} = \nu_{\text{local}} \left[1 + \frac{\mathbf{n} \cdot \mathbf{V}_\oplus}{c} + \mathcal{O}\left(\frac{1}{c^2}\right) \right]. \quad (\text{A.2})$$

We now apply (A.2) to, for example, the satellite to ground normalized frequency difference to obtain directly

$$\left(\frac{\nu_{SA} - \nu_A}{\nu_0} \right)_{\text{global}} = \left(\frac{\nu_{SA} - \nu_A}{\nu_0} \right)_{\text{local}} \left[1 + \frac{\mathbf{n}_{SA} \cdot \mathbf{V}_\oplus}{c} + \mathcal{O}\left(\frac{1}{c^2}\right) \right]. \quad (\text{A.3})$$

In the correction term we can replace $(\nu_{SA} - \nu_A)/\nu_0$ by the leading term of (41) which leads directly to (42).

Appendix B. Transformation of unit direction vectors

We use the transformation (23) with $T_0 = T_S$ (the emission time on the satellite) to transform the unit vector \mathbf{N}_{SA} . Working to only first order in V/c , Eq. (23) leads to

$$\mathbf{X}_S = \mathbf{x}_S + \mathbf{X}_\oplus, \quad \mathbf{X}_A = \mathbf{x}_A + \mathbf{X}_\oplus + \frac{R_{SA}}{c} \mathbf{V}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (\text{B.1})$$

where we have used $T_A - T_S = R_{SA}/c$. From (B.1) we obtain

$$R_{SA} = |\mathbf{X}_A - \mathbf{X}_S| = |\mathbf{x}_A - \mathbf{x}_S + \frac{R_{SA}}{c} \mathbf{V}_\oplus| = r_{SA} \left(1 + \frac{\mathbf{n}_{SA} \cdot \mathbf{V}_\oplus}{c} \right) + \mathcal{O}\left(\frac{1}{c^2}\right), \quad (\text{B.2})$$

where $\mathbf{r}_{SA} = \mathbf{x}_A - \mathbf{x}_S$ and $\mathbf{n}_{SA} = \mathbf{r}_{SA}/r_{SA}$. Using (B.1) and (B.2) then directly leads to

$$\mathbf{N}_{SA} = \frac{\mathbf{X}_A - \mathbf{X}_S}{R_{SA}} = \mathbf{n}_{SA} \left(1 - \frac{\mathbf{n}_{SA} \cdot \mathbf{V}_\oplus}{c} \right) + \frac{1}{c} \mathbf{V}_\oplus + \mathcal{O}\left(\frac{1}{c^2}\right). \quad (\text{B.3})$$

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